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ON THE CHARACTERISTICS OF DIFFERENTIAL EQUATIONS*

BY E. R. HEDRICK

INTRODUCTION

THE importance of the theory of characteristics in the study of differential equations is well known to all who are interested in that subject. The ordinary developments† are, however, somewhat lacking in rigor. It is the purpose of this paper to present, in somewhat altered form, a new method for the introduction and treatment of characteristics, devised by Hilbert and given by him in lectures at Göttingen 1900–1901.

In developing the method, no previous knowledge of the theory will be assumed, but attention will be called to well known results when they appear.

Besides the new introduction thus proposed, a few theorems new to the theory will be given.‡

PART I. PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

1. The Cauchy-Kowalewski Theorem. The process to be developed depends directly on a theorem due to Cauchy§ and Kowalewski,‖ which will be assumed as known in the following form:

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* This paper includes two papers read before the American Mathematical Society: (1) "On the Characteristics of Partial Differential Equations," read Dec. 28, 1901; (2) "On the Integral Curves and Strips of Partial Differential Equations," read Feb. 28, 1903.

† See Goursat, *Éq. aux dér. part. 1 et 2 ordres;* Fricke, *Functionentheoretische Vorlesungen;* von Weber, *Encyc. d. Math. Wiss.* II, A, 5; Monge, *Applications de l'analyse à la géométrie*, 1795 (Edition Liouville 1850); etc.

‡ The theorems of §§ 2, 3, 5, 6 of Part I, and §§ 2, 3 of Part II are for the most part due to Hilbert; but some of the alterations made are essential. The work of §§ 2, 3, in each Part is somewhat altered; that of § 6, Part I is somewhat generalized, and is extended in §5, Part II. §4, Part I, and § 4, Part II, together with those parts of the previous sections which lead up to these sections, were not given by Hilbert, and are believed to be new.

§ *Oeuvres complètes de Cauchy*, 1re serie, vol. 7.

‖ *Crelle*, vol. 80.

Theorem. Let us consider any partial differential equation of the first order in one dependent and two independent variables:

$$(1) \qquad F(x, y, z, p, q) = 0,$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$; and let this equation be solved for p:

$$(2) \qquad p = f(x, y, z, q).$$

Then there exists one and only one analytic solution

$$z = \nu(x, y)$$

($i.$ $e.$ one and only one analytic function of x and y which when substituted for z renders (2) an identity in x and y) *which satisfies the boundary condition*

$$(3) \qquad z]_{x=x_0} = \nu(x_0, y) = \phi(y),$$

where $\phi(y)$ *is a preassigned analytic function of* y *near* $y = y_0$; *provided that* $f(x, y, z, q)$ *is a single valued analytic function of* x, y, z, q ($i.$ $e.$ developable in absolutely convergent power series by Taylor's theorem) *in the neighborhood of the point* $x = x_0$, $y = y_0$, $z = z_0 = \nu(x_0, y_0) = \phi(y_0)$, $q = q_0 = \partial\nu(x_0, y_0)/\partial y_0 = \phi'(y_0)$, *and that* $\nu(x, y)$ *is analytic** *near* $x = x_0$, $y = y_0$.

The equation (1) can be put in the form (2), and the function $f(x, y, z, q)$ will have the desired properties mentioned above, if $F(x, y, z, p, q)$ is an analytic function of x, y, z, p, q near the point $x = x_0$, $y = y_0$, $z = z_0$, $p = p_0$, $q = q_0$ where $F(x_0, y_0, z_0, p_0, q_0) = 0$, and if further $\partial F/\partial p \neq 0$ at this point. In this case (2) is called the *normal form* of the differential equation (1). But the solution (2) may be many valued, and in such case the theorem will hold for each single valued solution separately. Corresponding to each single valued solution of the form (2) there will then exist one and only one solution $z = \nu(x, y)$ of (1) which passes through the curve

$$\begin{cases} z = \phi(y), \\ x = x_0, \end{cases}$$

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* This requirement that the solution be analytic is not necessary: see E. R. Hedrick, *Dissertation*, Göttingen, 1901, p. 19.

in the interval $y_1 < y < y_2$, provided the function $F(x, y, z, p, q)$ is analytic and $\partial F/\partial p \neq 0$ in this interval, for the values of z, p, q given by $z = \phi(y)$, $q = \phi'(y)$, $q = \psi(y)$, where $F[x_0, y, \phi(y), \phi'(y), \psi(y)] = 0$; and this solution $\nu(x, y)$ will be analytic in the interval for sufficiently small values of $|x - x_0|$.*

Geometrically the theorem then states that one and only one analytic integral surface of (1), corresponding to each single valued solution of the form (2), can be found, upon which lies a given analytic curve in a plane parallel to the YOZ plane, provided F is analytic and $\partial F/\partial p \neq 0$ for the values of x, y, z along the curve and the corresponding values of p and q given by (1) and by the equation of the tangent to the curve.

Similar statements of course hold on interchange throughout of x and y, and simultaneous interchange of p and q.

By a suitable transformation it is clear that, *in general*, any curve in space would uniquely determine a corresponding integral surface. This would not follow from the above theorem, however, in case the transformation which carries the given curve into the curve (3) transforms the equation (1) into an equation which cannot be thrown into the normal form (2). It is precisely this consideration, interesting for its own sake, which will eventually lead us to the characteristics themselves, and to the so-called integral curves. We shall find that the curves (which we shall call *integral curves*) for which the Cauchy-Kowalewski process may fail, in the above sense, are divided into two sorts, in general: (1) those which lie on integral surfaces, which we shall call *characteristic curves*; (2) those which do not lie on any integral surface. We shall then see that in general several integral surfaces pass through a characteristic curve. Hence the Cauchy-Kowalewski theorem is actually untrue for any of the integral curves, including both the sorts mentioned above, for in neither case does the curve determine an integral surface through itself, uniquely.

We proceed at once to the determination of the characteristic curves. We shall find first the characteristic curves on a given integral surface, and later all the characteristic curves in space, *i. e.*, all those curves which lie on integral surfaces and for which the Cauchy-Kowalewski process may fail, in the above sense.

After having established some of the principal properties of characteristic curves, we shall determine the integral curves in general; and then state a few general theorems based upon the foregoing results.

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* See preceding footnote.

2. Failure of the Cauchy-Kowalewski Process. Let us then first assume given an integral surface

$$z = \nu(x, y), \tag{5}$$

where $\nu(x, y)$ is a known analytic function of x and y which satisfies the equation (1). By the above reasoning we see that any curve on this surface will, in general, determine an integral surface uniquely. We wish to find first the characteristic curves on this surface, *i. e.* the curves for which this conclusion cannot be drawn from the Cauchy-Kowalewski theorem, in the above sense. Aside from the curves already considered in §1, the analytic curves on the surface (5) may be represented by the equations

$$\begin{cases} y = \lambda(x), \\ z = \nu(x, y), \end{cases} \quad \text{or} \quad \begin{cases} y = \lambda(x), \\ z = \nu[x, \lambda(x)] = \mu(x), \end{cases} \tag{6}$$

where $\lambda(x)$, and hence also $\mu(x)$, are analytic functions of x.

Let us now transform the equation (1) by the transformation

$$\begin{cases} x_1 = y - \lambda(x), \\ y_1 = x, \end{cases} \quad \text{or} \quad \begin{cases} x = y_1, \\ y = x_1 + \lambda(y_1), \end{cases} \tag{7}$$

in order to transform our boundary curve (6) into the form

$$\begin{cases} x_1 = 0, \\ z = \mu(y_1), \end{cases} \tag{8}$$

which is more suitable for the application of the Cauchy-Kowalewski theorem. In the new variables x_1, y_1 the function z will have derivatives $\partial z/\partial x_1$, $\partial z/\partial y_1$, which we shall call p_1, q_1 respectively. From (7) we have at once:

$$\begin{cases} p = -p_1 \lambda'(x) + q_1, \\ q = p_1, \end{cases} \quad \text{or} \quad \begin{cases} p_1 = q, \\ q_1 = p + q\lambda'(y_1), \end{cases} \tag{9}$$

where $\lambda'(x) = d\lambda(x)/dx$. Hence the given differential equation (1) becomes

$$(10) \qquad F[y_1, x_1 + \lambda(y_1), z, q_1 - p_1 \lambda'(x), p_1] = 0.$$

The Cauchy-Kowalewski theorem will then be applicable to the curve (8), provided F is analytic and $\partial F/\partial p_1 \neq 0$ for each set of values of x_1, y_1, z_1, p_1, q_1 in a certain interval given by $x_1 = 0$, $y_1 = y_1$, $z = \mu(y_1)$, $p_1 = \psi(y_1)$, $q_1 = \mu'(y_1) = d\mu(y_1)/dy_1$, where

$$F[y_1, \ 0 + \lambda(y_1), \ \mu(y_1), \ \mu'(y_1) - \psi(y_1)\lambda'(y_1), \ \psi(y_1)] = 0.$$

The same result is evidently obtained by writing $\partial F/\partial p_1$ as a function of x, y, z, p, q and substituting $x(= y_1) = x$, $y(= x_1 + \lambda y_1) = 0 + \lambda(x)$, $z = \mu(x)$, $p[= q_1 - p_1 \lambda'(x)] = \phi_1(x)$, $q(= p_1) = \phi_2(x)$, where

$$F[x, \ \lambda(x), \ \mu(x), \ \phi_1(x), \ \phi_2(x)] = 0$$

and where $\phi_1(x) = \mu'(x) - \phi_2(x)\lambda'(x)$. But again this same result is obtained by substituting $z = \nu(x, y)$, $p = \nu_x(x, y)$, $q = \nu_y(x, y)$ — where $\nu_x(x, y) = \partial\nu(x, y)/\partial x$ and $\nu_y(x, y) = \partial\nu(x, y)/\partial y$ — and then $y = \lambda(x)$. For $\nu[x, \lambda(x)] = \mu(x)$ and the quantities substituted for p and q satisfy the relations for $\phi_1(x)$ and $\phi_2(x)$, since

$$\mu'(x) = \nu_x[x, \lambda(x)] + \nu_y[x, \lambda(x)] \cdot \lambda'(x)$$

and $F[x, y, \nu(x, y), \nu_x, y_y] = 0$ identically in x and y, because (5) is a solution of (1).

The curves on the surface (5) for which the Cauchy-Kowalewski process may fail, in the above sense, are then given by the equation:

$$(11) \qquad \frac{\partial F}{\partial p_1} \equiv -\frac{\partial F}{\partial p}\frac{d\lambda(x)}{dx} + \frac{\partial F}{\partial q} = 0,$$

where z, p, q are to be replaced by $\nu(x, y)$, $\nu_x(x, y)$, $\nu_y(x, y)$, respectively. For, by the above, this equation becomes an identity in x, for any curve on (5) for which the Cauchy-Kowalewski process might fail in the above sense, when these substitutions are made and then $\lambda(x)$ is substituted for y. But $\nu(x, y)$ is a known analytic function. Hence, after the substitution of $\nu(x, y)$

throughout for z, the equation (11) becomes an ordinary differential equation for $\lambda(x)$, *i. e.* for y as a function of x:

$$(11a) \qquad -F_p(x, y)\frac{dy}{dx} + F_q(x, y) = 0,$$

where F_p and F_q denote the functions of x and y which result from the substitution of $\nu(x, y)$ throughout for z in $\partial F/\partial p$ and $\partial F/\partial q$ respectively. The equation (11a) has one and only one solution corresponding to any preassigned set of values of x and y, in a region where the coefficients are analytic and $F_p \neq 0$; and this solution is an analytic function of x. Hence:

THEOREM. *There exists on the given surface* (5) *a one parameter family of curves of the type* (6), *satisfying* (11a), *one and only one of which passes through any given point of the surface, in a region where F_p and F_q are analytic and $F_p \neq 0$.*

For these curves the Cauchy-Kowalewski process fails, in the above sense; *i. e.*, we cannot assert that one of these curves determines uniquely the integral surface on which it lies. *That it actually does not, remains to be proved.* As in §2, we shall call these curves *characteristics* or *characteristic curves*, since by definition they lie on an integral surface. The equation (11a) will be recognized as the principal equation of the characteristic curves, as ordinarily derived.* It follows that *every integral surface is completely covered by characteristic curves, one and only one of which passes through any given point of the surface, provided F_p and F_q are analytic and $F_p \neq 0$ at that point.* And we see that unless F is not analytic near some point given by the substitution of $x = x$, $y = \lambda(x)$, $z = \nu[x, \lambda(x)]$, $p = \nu_x[x, \lambda(x)]$, $q = \nu_y[x, \lambda(x)]$ found above, the Cauchy-Kowalewski theorem directly applies. Excluding such exceptional points (where F is not analytic) we have the

THEOREM: *Any curve of the type* (6) *which lies on an integral surface and is not a characteristic curve, determines uniquely the integral surface on which it lies, under the above restriction.*

We have tacitly assumed that F_p does not vanish identically in x and y along the surface (5). If it does, we can easily modify our work so that x and y are interchanged unless F_q also vanishes identically. If now F_p and F_q both vanish identically along the given surface (5), the equation (11) is identically satisfied for any curve whatever on the surface, and the Cauchy-Kowalewski process may fail for every such curve.

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* See Goursat, *l. c.*, *1 ordre*, p. 115; Fricke, *l. c.*, p. 491; and others.

We are thus led to enquire into the nature of the surfaces obtained by eliminating p and q between the three equations

$$F = 0, \quad F_p = 0, \quad F_q = 0. \tag{12}$$

If, as a result of such elimination a surface $z = \nu(x, y)$ is found, which satisfies (1) and for which F_p and F_q both vanish identically in x and y, then such an integral surface must be excluded in the statement of the theorems above, and in fact *every curve on such a surface would satisfy the definition of a characteristic curve.* It is an advantage of the present method that our attention is called at once to such surfaces, for it turns out that they are precisely the so-called *singular solutions.*[*] We shall exclude them from our discussion for the present, however, in view of their evident peculiarity, in order that the statements of theorems may remain simple.

Likewise, if F_p vanishes at any point (x_0, y_0) of the surface (5), the theorems of this section hold true unless F_q vanishes at the same point. But if both F_p and F_q vanish at any point (x_0, y_0, z_0) of the surface (5), then we cannot assert that one and only one solution of (11a) passes through this point, and such points must be excluded from the discussion in the statement of the theorems. They will in general be points upon the envelope of the one parameter family of solutions of (11a), or in particular the point into which such an envelope may degenerate. Such points, or their locus, may be found by substituting $\nu(x, y)$ throughout for z in $\partial F/\partial p$ and $\partial F/\partial q$ and solving the resulting equations $F_p(x,y) = 0$, $F_q(x,y) = 0$ as simultaneous equations for x and y. This will give, in general, discrete points in the (x, y) plane, which are singular points for the surface (5), in the above sense.

It now remains to determine the characteristics without assuming $\nu(x, y)$ as known, and this we shall do in §3. On the other hand, it is clear that we have used the assumption that the curve (6) lies on the integral surface (5) only to show that the substitution of $\nu(x, y)$ for z and of $\nu_x[x, \lambda(x)]$ and $\nu_y[x, \lambda(x)]$ for p and q respectively was equivalent to the substitutions for z_1, p, q_1 which were to be made in F_{p_1}. We cannot make the above substitution, of course, when we do not assume that the curve for which the process is to fail, lies on some integral surface. We shall then return to this point in §4, and seek to determine the *integral curves* in general, *i. e.*, whether they lie on an integral surface or not. The characteristic curves are evidently also integral curves.

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[*] See Darboux, "Mémoire sur les solutions singulières," *Jour. sav. étr.*, vol. 27 (1880); Goursat, *l. c.*, *1 ordre*, p. 86; Forsyth, *Differential Equations*, p. 296; etc.

3. The Characteristic Curves in Space. We have defined a characteristic curve to be a curve which lies on some integral surface and for which the Cauchy-Kowalewski theorem may fail in the above sense. We shall now proceed to prove that the theorem actually does not hold, *i. e.*, that there are other integral surfaces through this same curve.

We shall first obtain the differential equations of the characteristic curves, which by definition lie on an integral surface, without assuming the integral surface to be given. Since $\nu(x, y)$ is a solution of (1), that equation will become an identity in x and y when $\nu(x, y)$ is substituted for z. Likewise the two functions dF/dx and dF/dy will vanish identically when $\nu(x, y)$ is substituted for z:

$$(13)\qquad \begin{cases} \dfrac{dF}{dx} \equiv F_p p_x + F_q q_x + F_z p + F_x = 0, \\ \dfrac{dF}{dy} \equiv F_p p_y + F_q q_y + F_z q + F_y = 0, \end{cases}$$

where $p_y = dp/dy = \partial^2 z/\partial x \partial y = \partial q/\partial x = q_x$, and so on.

We have also the identities

$$(14)\qquad \begin{cases} \dfrac{dz}{dx} = p + q\,\dfrac{dy}{dx}, \\ \dfrac{dp}{dx} = p_x + p_y\,\dfrac{dy}{dx}, \\ \dfrac{dq}{dx} = q_x + q_y\,\dfrac{dy}{dx}, \end{cases}$$

where $z = \nu(x, y)$, and $y = \lambda(x)$, any function.

The equations (1), (11), (13), (14) are satisfied identically in x, when $\nu(x, y)$ is substituted for z and then $\lambda(x)$ for y, where $\nu(x, y)$ is a solution of (1), and $\lambda(x)$ is a corresponding solution of (11); or, what is the same thing, these equations (1), (11), (13), (14) are satisfied identically in x by the equations $z = \mu(x)$, $y = \lambda(x)$, the equations of the desired characteristic curves. Hence (1), (11), (13), (14) are the differential equations of the characteristic curves. From them we can eliminate p_x, $p_y (= q_x)$, and q_y, so as to obtain the equations:

$$(15)\qquad \begin{cases} \dfrac{dy}{dx} = \dfrac{F_q}{F_p}, & \dfrac{dp}{dx} = -\dfrac{F_x + pF_z}{F_p}, \\ \dfrac{dz}{dx} = \dfrac{pF_p + qF_q}{F_p}, & \dfrac{dq}{dx} = -\dfrac{F_y + qF_z}{F_p}, \end{cases}$$

which, together with (1), must be satisfied by $y = \lambda(x)$, $z = \nu[x, \lambda(x)] = \mu(x)$, $p = \left.\dfrac{\partial \nu(x, y)}{\partial x}\right]_{y = \lambda(x)} = \phi_1(x)$, and $q = \left.\dfrac{\partial \nu(x, y)}{\partial y}\right]_{y = \lambda(x)} = \phi_2(x)$, say.

The equations (15) are then a set of four simultaneous ordinary differential equations for the determination of y, z, p, q as functions of x. The equation (1) must, however, also be satisfied, and hence one of the equations (15) (say the last), may be regarded as supernumerary, and the value of q (say) may be found from (1), after y, z, p have been determined by the first three of equations (15).

If, now, we think of x, y, z, p, q all as functions of a new auxiliary variable t, we shall have $\dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dy}{dt}$, etc., whence the equations (15) become:*

$$(16)\qquad \begin{cases} \dfrac{dx}{dt} = F_p, \qquad \dfrac{dy}{dt} = F_q, \qquad \dfrac{dz}{dt} = pF_p + qF_q, \\ \dfrac{dp}{dt} = -(F_x + pF_z), \qquad \dfrac{dq}{dt} = -(F_y + qF_z), \end{cases}$$

of which the first equation is merely a definition of t. The set (15) [or (16)] is seen at once to be the set of equations ordinarily obtained for characteristics.†

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* This form is also obtained if we exchange x and y in all our work. Or we might obtain it directly by altering the work slightly. It is convenient in avoiding an interchange of x and y when F_p vanishes.

† See Goursat, *l. c.*, *1 ordre*, p. 115; Fricke, *l. c.*, p. 491; etc.

By the Cauchy-Kowalewski theorem for ordinary equations we know that if five constants x_0, y_0, z_0, p_0, q_0 are given and if

$$x]_{t=0} = x_0; \quad y]_{t=0} = y_0; \quad z]_{t=0} = z_0; \quad p]_{t=0} = p_0; \quad q]_{t=0} = q_0$$

be assumed as initial values, then a set of solutions of the equations (16) is uniquely determined:*

$$(17) \qquad \begin{cases} x = f_1\ (t, x_0, y_0, z_0, p_0, q_0) \\ y = f_2\ (t, x_0, y_0, z_0, p_0, q_0) \\ z = f_3\ (t, x_0, y_0, z_0, p_0, q_0) \\ p = f_4\ (t, x_0, y_0, z_0, p_0, q_0) \\ q = f_5\ (t, x_0, y_0, z_0, p_0, q_0) \end{cases}$$

and the five functions so obtained are analytic functions of x_0, y_0, z_0, p_0, q_0 as well as of t; provided only that the right hand sides of the given equations [as is the case in equations (16)] are analytic functions of all the variables present, near the point x_0, y_0, z_0, p_0, q_0.

The first three of equations (17) define a curve in space. The last two define, at each point of this curve, a tangent plane to the curve, by (14). This configuration of a curve in space together with tangent planes affixed at every point, we shall call a *strip*. Such a strip is, by the above, uniquely determined as the solution of equations (16) [or (15)] when any point (x_0, y_0, z_0) and a plane (p_0, q_0) through that point are given. This configuration of a point together with a plane through it, is called a *surface element*.

Since for our purposes x, y, z, p and q must, when found as functions of t, render the equation (1) an identity in t, it will be necessary to assume that x_0, y_0, z_0, p_0, q_0 satisfy (1), since these are to be the values of x, y, z, p, and q corresponding to $t = 0$. We shall then have $F(x_0, y_0, z_0, p_0, q_0) = 0$, or the surface element $(x_0, y_0, z_0, p_0, q_0)$ lies on an integral surface. Let us then assume that x_0, y_0, z_0 are the coordinates of a point on some integral surface, and that p_0, q_0 are the directions of that integral surface, at (x_0, y_0, z_0). We have seen in §2 that there exists one and only one characteristic line, on the integral surface, through any point of the integral surface; and with it are associated the uniquely determined tangent planes which are also tangent to the given integral surface. Such a configuration of a characteristic line together with the appended tangent planes belonging to the integral surface we shall naturally call a *characteristic strip*. But we have just seen that through any surface element $(x_0, y_0, z_0, p_0, q_0)$ there goes one and only one strip satisfying the equations (16). But these equations merely express the condition for a characteristic line on some integral surface, together with the appended tangent planes uniquely determined at every point by this integral surface. For they are constructed out of equation (11) and the identities (13) and (14). From this follows directly the

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* See Picard, *Traité d'analyse*, vol. 2, p. 304; Fricke, *l. c.*, p. 435; etc.

THEOREM: *Any surface element* $(x_0, y_0, z_0, p_0, q_0)$ *lying on an integral surface uniquely determines a characteristic strip, which lies wholly on the surface.*

For, if it did not lie wholly on the surface, then the characteristic strip given by §2 would be a second characteristic strip through the given surface element and lying on the integral surface, the existence of which is impossible.

Or, if x_0, y_0, z_0, p_0, q_0 *be a set of values satisfying the given differential equation* (1), *then the set of solutions of* (16) [or (15)]: x, y, z, p, q, *given by* (17), *satisfy the equation* (1) *identically in t.**

There is a three parameter family of characteristic strips determined by the equations (16) [or (15)]. For, of the five constants x_0, y_0, z_0, p_0, q_0 in (17), one is determined by the relation $F(x_0, y_0, z_0, p_0, q_0) = 0$, and we get the same characteristic strip for each surface element of the one parameter family of surface elements lying along it. This leaves a three parameter family of characteristic strips in all space.

This three parameter family of strips will in general lie along a three parameter family of characteristic curves, given by the first three of equations (17). But it may happen that several characteristic strips lie along the same characteristic curve, so that there may be less than three parameters entering into the equations of the characteristic curves. In particular, for an equation of the form

$$A(x, y, z)p + B(x, y, z,)q = C(x, y, z)$$

the first two of equations (15) [or the first three of (16)] do not involve p or q, and can be solved separately. In this case, then, the first three of equations

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* The analytic proof is also immediate. See, *e. g.*, Fricke, *l. c.*, p. 433.

(17) involve only two essential parameters, and hence there is only a two parameter family of characteristic curves. If we solve all of equations (15) [or (16)] we shall find as before a three parameter family of characteristic strips, which are then arranged along the two parameter family of curves in such a way that through each curve there passes, in general, a one parameter family of strips. In general, then, any plane tangent to one of the curves determines a characteristic strip through the curve.

In general there is a three parameter family of characteristic curves with one characteristic strip along each curve, but in particular some of these may unite. For an equation linear in p and q, there is thus only a two parameter family of curves with a one parameter family of characteristic strips along each curve.

It is easily seen that an infinity of integral surfaces pass through any given surface element which satisfies (1). For instance, all curves, not integral curves, tangent to the element, determine uniquely integral surfaces through themselves, by the Cauchy-Kowalewski theorem. By the above, however, the unique characteristic strip determined by the given surface element lies on any integral surface through that element. Hence we have the following

THEOREM: *The characteristic curves, defined as those curves on integral surfaces for which the Cauchy-Kowalewski process may fail, actually do not determine the integral surfaces on which they lie. There are in fact an infinity of integral surfaces through every characteristic curve, and these surfaces are all tangent along this curve, at least provided they are tangent at one point.* For, the characteristic strip determined by their one common surface element, lies on each integral surface. The condition stated in the last line is not necessary, except when, as for linear equations, more than one strip lies along each characteristic line.

Let us now consider the curve (3); for which the Cauchy-Kowalewski theorem was stated in §1:

$$(18) \qquad \begin{cases} z = \phi(y) \\ x = x_0 \end{cases}$$

where $\phi(y)$ is analytic. This curve uniquely determines an integral surface of (1) if F is analytic and $\partial F/\partial p \neq 0$ in the interval considered for $x = x_0$, $y = y$, $z = \phi(y)$, $p = \psi(y)$, $q = \phi'(y)$, where $F[x_0, y, \phi(y), \psi(y), \phi'(y)] = 0$.

The curve (18) is not of the form (6), but we see that the Cauchy-Kowalewski theorem can fail only when $F_p = 0$ for the above substitutions, *i. e.*, for the functions $\phi(y)$ determined by (11), after dy/dx is replaced by $(dx/dy)^{-1}$; for since x_0 is constant, $dx/dy = 0$ for this curve. Hence any curve of the form (18) as well as of the form (6), for which the Cauchy-Kowalewski theorem fails, must satisfy (11), if, as above, we exclude points of the curve for which F is not analytic. If $F_p \equiv 1$, *i. e.*, if the equation is written in the form (2), it is clear that the only exceptional curves of the class (18) are those for which F(*i. e.* in this case f) is not analytic, which is precisely a restatement of the Cauchy-Kowalewski theorem. If $F_p \equiv 0$ in x, y, z, p, q, that is, if p does not occur in F, it is clear that every curve of the type (18) would be such an exceptional curve.

If $F_p \equiv 0$, however, we can interchange x and y in all that follows, unless F_q also vanishes identically in x, y, z, p, q, in which case (1) ceases to be a differential equation.

Assuming then that $F_p \not\equiv 0$, excluding the curves (18) which satisfy (11), and restricting ourselves, for the present, to an interval where F is analytic along the curve, it follows directly from the Cauchy-Kowalewski theorem that the curve (18) determines uniquely an integral surface. At any point $x_0, y_0, z_0 = \psi(y_0)$ of this curve we shall have

$$q_0 = \left.\frac{d\psi(y)}{dy}\right]_{y=y_0}$$

while p_0 is determined from the relation $F(x_0, y_0, z_0, p_0, q_0) = 0$. Hence a surface element is determined at each point of the curve. This surface element, in turn, determines uniquely a characteristic strip through this point, and this characteristic strip lies on the integral surface determined by the given curve. If now we let the point move along the curve, the corresponding characteristic curve (with its strip) evidently sweeps out the integral surface. The analytic statement consists in setting

$$(19) \qquad \begin{cases} x_0 = x_0, \text{ constant}; \\ y_0 = \eta\ , \text{ variable}; \\ z_0 = \psi(\eta), \text{ arbitrary}; \end{cases} \qquad \begin{aligned} p_0 &= \phi(\eta), \\ q_0 &= \left.\frac{d\psi(y)}{d(y)}\right]_{y=\eta} = \psi'(\eta)\ , \end{aligned}$$

where $\phi(\eta)$ is determined by the identity in η:

$$F[x_0, \eta, \psi(\eta), \phi(\eta), \psi'(\eta)] \equiv 0.$$

The single characteristic *curve* given by $y_0 = \eta$ for any constant value of η is, from (17),

$$(20) \qquad \begin{cases} x = f_1[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \\ y = f_2[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \\ z = f_3[t, x_0, \eta, \psi(\eta), p_0, \psi'(\eta)], \end{cases}$$

where p_0 is determined as a function of $\eta, \psi(\eta)$, and $\psi'(\eta)$, as above. The two similar equations derived from the last two of equations (17) define the directions of a tangent plane at each point of this curve. The equations (20), for *variable* η, for any function ψ, define a single *surface*, and this is an integral surface, by the above. For every choice of ψ we get one such surface, where t and η are parameters, p_0 having been obtained as above as a function of η, $\psi(\eta)$, and $\psi'(\eta)$; and these are all the regular integral surfaces of the given equation (1). Hence we obtain the following

THEOREM: *If we can obtain the characteristics, as the solution, in the form* (17), *of equations* (16) [*or* (15)], *we can at once write down the general solution of the given equation* (1) *in the form* (20), *involving in general on arbitrary function and its derivative.*

The importance of characteristics in the study of differential equations is well illustrated by the theorem just stated. But they do even more than aiding directly in the solution, by making clear the geometrical meaning of all parts of the subject. It is not the purpose of this paper to develop further the extensive theories connected with characteristic lines and strips, the ordinary theory being the same as ours would be from this point on.* Indeed we have passed over already with only short geometric proofs, two theorems (p. 134 and p. 132), of which rigorous analytical proofs can be given, those given by Fricke (*l. c.*, p. 433) being satisfactory.† It can be shown, in fact, that *the equations* (20) *represent a solution even when* $\phi(\eta)$ *is merely a continuous function which has a continuous first derivative*, though the proof will be essentially different from that sketched above, in that we cannot then infer *from the Cauchy-Kowalewski theorem* the *existence* of an integral surface through the given curve.*

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* See, *e. g.*, Goursat, *l. c.*, *1 ordre*, Chap. IV.

† See also Goursat, *l. c.*, *1 ordre*, p. 110.

It is now easy to show that *every analytic partial differential equation of the first order has non-analytic solutions.* For if $\psi(\eta)$ be taken non-analytic, it follows easily that the surface determined by (20) is non-analytic near this curve, in a two dimensional region. The details of the proof are given in the writer's thesis.†

4. The Integral Curves. Geometrical Interpretation. While our chief purpose has been attained in the discovery of the equations for the characteristics, it is interesting to discover the equations defining all those curves in space for which the Cauchy-Kowalewski theorem may not hold. These curves have been called (§1) the *integral curves.*

We shall exclude from the discussion, as above, on any curve,

$$\text{(6a)} \qquad \begin{cases} y = \lambda(x), \\ z = \mu(x), \end{cases}$$

any point $x_0 = x_0$, $y_0 = \lambda(x_0)$, $z_0 = \mu(x_0)$, if F is non-analytic about the point x_0, y_0, z_0, p_0, q_0, where $F(x_0, y_0, z_0, p_0, q_0,) = 0$ and $\mu'(x_0) = p_0 + q_0 \lambda'(x_0)$.

We shall then have to consider merely the possibility of the vanishing of $\partial F/\partial p$ for this curve, *i. e.* for the values $x = x$, $y = \lambda(x)$, $z = \mu(x)$, $p = \phi_1(x)$, $q = \phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ satisfy the equations $F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x)] = 0$, and $\mu'(x) = \phi_1(x) + \phi_2(x)\lambda'(x)$. For, as remarked at the close of §2, these are exactly the substitutions we must make when we do not assume (as in §2) that the curve (6*a*) lies on an integral surface. If $\lambda(x)$ and $\mu(x)$ are unknown, p and q in (11) may then be replaced by the values found by solving the equations (1) and

$$\text{(14a)} \qquad \frac{d\mu(x)}{dx} = p + q\frac{d\lambda(x)}{dx}$$

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* The proof given by Fricke, *l. c.*, p. 443, is immediately extensible to this case.

† See Hedrick, *l. c.*, pp. 20–25. This theorem is not extensible to systems of partial differential equations of the first order; see Hedrick, *l. c.*, p. 25.

as simultaneous equations for p and q. Any functions $\lambda(x)$ and $\mu(x)$ which satisfy the resulting equation

$$M\left(x,\, y,\, z,\, \frac{dz}{dx},\, \frac{dy}{dx}\right) = 0, \tag{11b}$$

when substituted for y and z respectively, will then evidently define an integral curve of the form (6a). The equation (11b) is due to Monge, and is called the *Monge equation.* In general it will be determined by eliminating p and q between the equations (1), (11), and (14a), written in the form

$$\frac{dz}{dx} = p + q\,\frac{dy}{dx}.$$

The equation (11b) is a single ordinary differential equation for the determination of z and y as functions of x. In general its solutions cannot be expressed by a finite number of parameters, for if any analytic function of x be substituted for z, for instance, there will still be in general a one parameter family of solutions for y, which, with the above arbitrary value of z, satisfy the equation. Among the numerous solutions will be included, of course, the characteristic curves, by definition. And it is otherwise evident that any solution of (15) will also satisfy (11b).

The values of p and q corresponding to given values of x, y, and z, which we have found it necessary to substitute here (and in §2) are found by solving (1) and (14a) as simultaneous equations. Now (1) constitutes for any fixed point (x, y, z) one relation between p and q, and defines at this point a one parameter family of planes (p, q) which in general envelop a cone whose vertex is in the point (x, y, z). It is clear that these are precisely the possible tangent planes to any integral surface through (x, y, z). Hence any integral surface must at every point be tangent to the cone through that point; and this condition is also sufficient. This *cone field* is then the exact geometrical counterpart of the differential equation. The conditions (11) and (14a) mean geometrically that the integral curve is tangent at every point to the cone at that point.* We obtained the characteristics however, under these same conditions, with the additional requirement that the curve should lie on an integral surface. Hence we have the

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* This geometrical analogy between integral surfaces and integral curves constitutes the justification of the latter terminology.

THEOREM. *The integral curves are those curves in space which are tangent at every point to the cone field determined by the equation* (1) ; *and the characteristic curves are those integral curves which lie on integral surfaces.* It is this geometrical fact which is usually made the basis of the whole theory.

For the linear equation

$$A(x, y, z,)\,p + B(x, y, z,)q = C(x, y, z)$$

however, the general propositions are materially altered. For (11) becomes

$$A(x, y, z)\,\frac{dy}{dx} = B(x, y, z), \tag{11c}$$

which does not involve p or q. It is therefore unnecessary to make the substitutions which were made in general, and any curve of the form (6a) which satisfies (11c) is an integral curve. If for y any function of x, say $\lambda(x)$, is substituted the equation (11c) will in general define a cylinder perpendicular to the plane of XOZ. *There is, therefore in general, one integral curve whose projection on the* XOY *plane is an arbitrarily assigned curve.* Or, if for z any function of x and y be substituted, the equation (11c) will reduce to an ordinary differential equation for y. *On any surface in space there is then a one parameter family of integral curves, which completely cover the surface. Of these, those which lie on integral surfaces are characteristic curves* ; and these are, by §3, a two parameter family in space given by the first two of equations (14), of which the first coincides with (11c).

If, however A and B do not involve z, the facts are again changed. For then (11c) does not involve z, and hence becomes a single ordinary equation for y as a function of x. If $y = \lambda(x)$ satisfies this equation, the curve (6a) is an integral curve. *Hence, if A and B do not involve z, there is a one parameter family of curves in the* XOY *plane, such that any curve whose projection on this plane is a curve of this family is an integral curve. Of these curves, those which lie on integral surfaces are characteristic curves*; and these are given by solving the second of equations (15) :

$$A(x, y)\,\frac{dz}{dx} = C(x, y, z)$$

for z as a function of x, after replacing y by any solution $\lambda(x)$ of (11c).

It is easy to see that (11c) gives the correct results not only for curves of the form (6a) but also for any curve of the form (18).

It is evident in general that no integral surface will pass through an integral curve which is not a characteristic curve. And we have seen that a characteristic curve does not determine an integral surface uniquely. Hence:

THEOREM. *No integral curve determines uniquely an integral surface, there being more than one integral surface through any characteristic curve, and none at all through any other integral curve.*

Let us suppose the integral curves determined, in the form (6a) [or (18)]. If we consider any other curve, C, of the form (6a) [or (18)], it is clear that the Cauchy-Kowalewski theorem will hold, and that C will uniquely determine an analytic integral surface, for any portion of C for which F is analytic, and for which (11) does not hold at any point, when $\lambda(x)$, $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$ are substituted for y, z, p, q, as above. But (11) can hold only at isolated points on C, since $\lambda(x)$ and $\mu(x)$ are supposed analytic, and C would coincide with an integral curve if (11) were satisfied at a set of points of C which cluster about any point of C. If we exclude these *isolated* points on C, for which (11) holds, any remaining portion of C determines uniquely an analytic integral surface through itself, provided F is analytic along this portion of C. The points excluded are, geometrically, the points at which C is tangent to an integral curve, since $\lambda'(x)$ and $\mu'(x)$ are used in the above substitution to determine $\phi_1(x)$ and $\phi_2(x)$. And it is clear that any constant value of x, y, z, $\lambda'(x)$, $\mu'(x)$ which satisfy (11b) [or the corresponding equations from which (11b) is derived] determine one or more solutions of (11b); or of (11) under the above substitution.

If C lies on an analytic integral surface, it surely determines that integral surface uniquely; for each portion of it, between any two points which we have excluded, determines a portion of this same integral surface, uniquely. Hence, since the intervals about the excluded points may be made arbitrarily small, the whole curve C determines the whole surface uniquely. The only other cases which can arise are those in which C uniquely determines an integral surface, which is analytic along C except at the excluded points, and which is analytic in general except near an excluded point of C; or where the portions of C lying between excluded points determine pieces of *different* analytic integral surfaces. We may then state the

THEOREM: *Any curve C in space, of the form* (6a) *or* (18), *which is not an integral curve, determines uniquely an integral surface which is analytic* *and which passes through the given curve, for any portion of* C *in which no point lies at which* C *is tangent to an integral curve, or at which* $F(x, y, z, p, q)$ *is non-analytic: the point* x_0, y_0, z_0, p_0, q_0 *of the curve* C *being given by* $x = x$, $y = \lambda(x)$, $z = \mu(x)$, $p = \phi_1(x)$, $q = \phi_2(x)$, *where* $F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x)] = 0$ *and where* $\mu'(x) = \phi_1(x) + \phi_2(x)\lambda'(x)$, *for a curve of the form* (6*a*); *or by* $x = x_0$, $y = y$, $z = \phi(y)$, $p = \psi(y)$, $q = \psi'(y)$, *where* $F[x_0, y, \phi(y), \psi(y), \phi'(y)] = 0$, *for a curve of the form* (18).

5. Possible Families of Characteristics. It is easily (and usually)* shown that if

$$z = \phi(x, y, a, b), \tag{24}$$

represents any two parameter family of solutions† of (1), then the general solution of (1) is given by

$$\begin{cases} z = \phi[x, y, a, w(a)], \\ 0 = \phi_a + \phi_w w'(a), \end{cases} \tag{25}$$

where $w(a)$ is an arbitrary function; and the characteristic curves are given by

$$\begin{cases} z = \phi(x, y, a, b), \\ 0 = \phi_a(x, y, a, b) + c\phi_b(x, y, a, b), \end{cases} \tag{26}$$

as we see from equations (21) and (22).

We have seen that there is in general a three parameter family of characteristics of any partial differential equation of the first order. Let us now inquire if any three parameter family of curves may be the family of characteristic curves of some such differential equation. Let the equations

$$\begin{cases} h(x, y, z, a, b, c) = 0, \\ g(x, y, z, a, b, c) = 0 \end{cases} \tag{27}$$

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* See *e. g.*, Forsyth, *Differential equations*, p. 294; Goursat, *l. c.*, *1 ordre*, Chap. V; etc.

† The necessary and sufficient condition that a and b in (24) be independent parameters is that

$$\frac{\partial^2\phi}{\partial x\partial a}\frac{\partial\phi}{\partial b} - \frac{\partial^2\phi}{\partial x\partial b}\frac{\partial\phi}{\partial a} \text{ and } \frac{\partial^2\phi}{\partial y\partial a}\frac{\partial\phi}{\partial b} - \frac{\partial^2\phi}{\partial y\partial b}\cdot\frac{\partial\phi}{\partial a}$$

do not both vanish identically.

be the equations of the three parameter family considered. Solving for z and c, as is in general possible, we have

$$(28)\qquad \begin{cases} z = k(x,\ y,\ a,\ b), \\ c = l\,(x,\ y,\ a,\ b), \end{cases}$$

in which form we may suppose the family given. If this family is to coincide with (26) we must have

$$\frac{\partial k(x,\ y,\ a,\ b)}{\partial a} + l\,(x,\ y,\ a,\ b)\,\frac{\partial k(x,\ y,\ a,\ b)}{\partial b} = 0.$$

The general solution of this differential equation for k we know to be

$$(29)\qquad k = w(k_0,\ x,\ y),$$

where $k_0 = k_0(x,\ y,\ a,\ b)$ is any particular solution and w is an arbitrary function. *Hence the family* (27) *is by no means arbitrary. But if the projections of the curves* (27) *on the* XY *plane, i.e., the second of equations* (28), *be given, then there is a whole class of differential equations of the first order which have characteristics with these given projections; and these equations go over into one another by the transformation* $z' = w(z,\ x,\ y)$; *i. e., the differential equation, the solutions, and the characteristic curves are carried into each other by this transformation.*

Again, if we eliminate p and q between the equation (1) and the first two of equations (15), so as to get the Monge equations:

$$(30)\qquad M\left(\frac{dy}{dx},\ \frac{dz}{dx},\ x,\ y,\ z\right) = 0,$$

then we know that the characteristic curves of the equation (1) are among the solutions of this Monge equation, which has in addition infinitely many more solutions. Given now a three parameter family of solutions of (30);

$$\begin{cases} y = \phi_1(x,\ c_1,\ c_2,\ c_3), \\ z = \phi_2(x,\ c_1,\ c_2,\ c_3), \end{cases}$$

which are *not* characteristics of (1), it follows readily that the Monge equation and the differential equation (1) itself are each uniquely determined by this three parameter family of integral curves. It follows that they are not integral curves for any other differential equation than (1), and hence certainly not characteristic curves of any other equation than (1). But they are not characteristic curves for the equation (1), by hypothesis. Hence they are not characteristic curves for any differential equation of the first order. *Hence, any three parameter family of integral curves which are not characteristic curves of the corresponding partial differential equation of the first order, are not the characteristic curves of any such equation.* We also see that the same three parameter family of curves cannot be the family of characteristic curves of two distinct partial differential equations of the first order; for, regarded as integral curves, they determine uniquely the differential equation. An instructive geometrical interpretation of these results is immediate.

7. Connection with the Calculus of Variations. The characteristics of a differential equation may be defined as the extremals of a certain problem of the calculus of variations. For, let us consider the problem* of rendering the following integral an extremum (maximum or minimum):

$$(31) \qquad I = \int_{t_0}^{t_1} \frac{dz}{dt}\, dt,$$

under the auxiliary conditions:

$$(32) \qquad z' = \frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} \equiv px' + qy',$$

where z' denotes dz/dt, and so on; and

$$(33) \qquad F(x, y, z, p, q) = 0.$$

This problem is substantially to find that strip—equation (32) requires that the tangent plane be tangent to the curve—which is composed of integral surface elements (equation (33)), the steepness of which is an extremum.

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[*] This problem is not reducible to a problem in ordinary maxima and minima because the auxiliary conditions involve x' and y'.

The first necessary conditions which must be satisfied by a solution of the above problem are the same as those for the problem of rendering the following integral an extremum :*

$$(34)\qquad I' = \int_{t_0}^{t_1}\Big\{z' + \lambda(z' - px' - qy') + \mu F(x, y, z, p, q)\Big\}dt,$$

under no auxiliary conditions; where λ and μ are unknown functions of t. These conditions, commonly known as Lagrange's, are :

$$(35)\qquad \begin{cases} d(-\lambda p)/dt = \mu F_x, & 0 = -\lambda x' + \mu F_p, \\ d(-\lambda q)/dt = \mu F_y, & 0 = -\lambda y' + \mu F_q, \\ d(\lambda)/dt = \mu F_z, & z - px' + qy' = 0, \\ \qquad\qquad F(x, y, z, p, q) = 0. & \end{cases}$$

If now we eliminate λ, $d\lambda/dt$ and μ from these equations, we get at once, as necessary conditions for the given problem :

$$(36)\qquad \begin{cases} \dfrac{dy}{dx} = \dfrac{F_q}{F_p}, & \dfrac{dp}{dx} = -\dfrac{F_x + pF_z}{F_p}, \\ \dfrac{dz}{dx} = \dfrac{pF_p + qF_q}{F_p}, & \dfrac{dq}{dx} = -\dfrac{F_y + qF_z}{F_p}, \\ \qquad\qquad F(x, y, z, p, q) = 0. & \end{cases}$$

But these are precisely the equations (5) of the characteristic strips of the differential equation (1). Regarded as necessary conditions for the given problem of the calculus of variations, their solutions are called the *extremals* of the given problem. We have then the

THEOREM : *The characteristic strips of any partial differential equation (1) of the first order, are the extremals of the problem of the calculus of variations stated at the head of this section. Or, the characteristic strips are those*

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* The same results may be obtained by replacing z' in (31) directly by its value from (32) ; then replacing p[or q] by its value obtained by solving (33) ; and computing the necessary conditions for the resulting integral. For the statements here made see Kneser, *Lehrbuch der Variationsrechnung*, p. 20 and p. 117 ; Pascal, *Variationsrechnung* (German translation by Schepp), p. 34 and p. 44 ; Mayer, *Math. Ann.*, vol. 26 (1885), p. 74.

strips which satisfy the given differential equation, the steepness of which is an extremum.†

It seems that this connection between the calculus of variations and the theory of characteristics may be of value to one or to both of these theories. If, for instance, we seek the common characteristics of two given differential equations,

$$F(x, y, z, p, q) = 0, \tag{37}$$
$$G(x, y, z, p, q) = 0, \tag{38}$$

in order to find their common solutions, in case any such exist, the equations of the common characteristics are evidently given by the first necessary conditions of the problem of rendering I in (31) an extremum, under the auxiliary conditions (32),(33) [or (37)], and (38).

Let us suppose the equations (37) [or (33)] and (38) to be solved for p and q:

$$p = \phi(x, y, z,), \tag{39}$$
$$q = \psi(x, y, z,), \tag{40}$$

in which form we may suppose (37) and (38) given. Let us then substitute these values of p and q in the equation (32), and the resulting value of z' in turn in the given integral (31). This latter becomes

$$I = \int_{x_0}^{x_1} \left\{ \phi(x, y, z) + \psi(x, y, z) y' \right\} dx, \tag{41}$$

with no auxiliary conditions; where now z is to be regarded as an unknown function of x and y. The first necessary condition for the existence of an extremum is then

$$\frac{d}{dx}(\psi) = \frac{\partial}{\partial y}(\phi + \psi y'), \tag{42}$$

or

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial\psi}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}\frac{dy}{dx} = \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y}\frac{dy}{dx},$$

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* This second, geometric, statement is not rigorous, since the sufficient conditions of the calculus of variations may not be satisfied.

which reduces to

$$\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial z}\phi=\frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}\psi \tag{43}$$

when we take (39) and (40) into account. But this is precisely the necessary and sufficient condition for the existence of common characteristics of the given equations. Such a result indicates that the matter of this paragraph may be useful in simplifying certain considerations in the theory of characteristics.

8. Conclusion. Extension to other Equations. It is evident that the whole work up to this point may be extended without difficulty to the case of a single partial differential equation in one dependent and several independent variables. Such an extension will indeed present nothing essentially new. But the extension to equations of the second (or higher) order (or to the practically identical case of a set of equations of the first order) is essentially different in important particulars, and will for that reason be treated in the second part of this paper. The reason for such a distinction is that the solution of any single partial differential equation of the first order may be reduced to the solution of a set of ordinary differential equations, as we have done in the case of equation (1). But the solution of a partial differential equation of the second (or higher) order, or a set of two (or more) partial differential equations of the first order in two (or more) independent variables, cannot, in general, be reduced to the solution of a set of ordinary differential equations. It is obvious that such a fact must seriously affect the extension of the theory of characteristics to equations of the second order, for it was by the aid of the characteristics that we succeeded in reducing the solution of the equation (1) to the solution of the set (15) of ordinary differential equations. There remains however a considerable analogy between the above work and that for an equation of the second order, and an attempt will be made to abbreviate the following work wherever possible.